Recall: $\mathcal{A}$ set of arithmetic functions

$f, g \in \mathcal{A}$, $f * g(n) = \sum_{d \mid n} f(d)\, g\left(\frac{n}{d}\right)$

$(\mathcal{A}, +, *)$ commutative $\mathbb{C}$-algebra with mult identity $e(n) = \begin{cases} 1, & n = 1 \\ 0, & \text{else} \end{cases}$

- multiplicative functions, $f(mn) = f(m) f(n)$ if $(m, n) = 1$

examples: e, $id(n) = n$, $\varepsilon(n) = \mathbb{1}(n) = 1$, $\tau = \varepsilon * \varepsilon$.

- $\varphi(n) = |(\mathbb{Z}/n\mathbb{Z})^*|$. (mult).

Pf: CRT: $(n_1, n_2) = 1 \Rightarrow \mathbb{Z}/(n_1 n_2)\mathbb{Z} \simeq (\mathbb{Z}/n_1\mathbb{Z}) \times (\mathbb{Z}/n_2\mathbb{Z})$ as rings.

$\varphi(p^l) = p^{l-1}(p-1)$.

Theorem: If $f, g \in \mathcal{A}$ multiplicative, then $f * g$ multiplicative.

Ex: $\tau = \varepsilon * \varepsilon$ mult, but not completely mult.

$\tau(4) = 3 \neq 4 = \tau(2)^2$.

Pf: Suppose $(n_1, n_2) = 1$.

$$(f * g)(n_1 n_2) = \sum_{d \mid n_1 n_2} f(d)\, g\left(\frac{n_1 n_2}{d}\right) =$$

$$= \sum_{\substack{d_1 \mid n_1 \\ d_2 \mid n_2}} f(d_1 d_2)\, g\left(\frac{n_1}{d_1} \cdot \frac{n_2}{d_2}\right)$$

$\{d \mid n_1 n_2\} \to \{d_1 \mid n_1\} \times \{d_2 \mid n_2\}$

$d_1 d_2 \leftarrow (d_1, d_2)$

$$= \sum_{d_1 | n_1} f(d_1)\, g\left(\frac{n_1}{d_1}\right) \sum_{d_2 | n_2} f(d_2)\, g\left(\frac{n_2}{d_2}\right)$$

$$= (f * g)(n_1) \cdot (f * g)(n_2). \quad \square$$

Units in $\mathcal{A}$

Theorem $f \in \mathcal{A}$ unit (has inverse in $\mathcal{A}$ wrt $*$) $\Longleftrightarrow f(1) \neq 0$.

Pf: "$\Longrightarrow$" Suppose $\exists\, g \in \mathcal{A}$ s.t. $f * g = e$.
Then $1 = e(1) = f(1)\, g(1) \to f(1) \neq 0$.

"$\Longleftarrow$" We inductively define $g \in \mathcal{A}$ s.t. $f * g = e$.
Set $g(1) = \frac{1}{f(1)}$.

Suppose $g(1), \dots, g(n-1)$ already defined. Set

$$g(n) := -\frac{1}{f(1)} \sum_{d | n,\, d > 1} f(d)\, g\left(\frac{n}{d}\right).$$

Then $f * g(n) = e(n)$, $\forall n \in \mathbb{N}$. (easy exercise). $\square$

Rmk: We denote the inverse of f by f^{-1}.
If the inverse exists, it is unique (follows from the proof)

Proposition: If $f \in \mathcal{A}^\times$ multiplicative, then so is f^{-1}.

Proof: Let $g \in \mathcal{A}$ multiplicative defined by

$$g(p^\ell) = f^{-1}(p^\ell), \; p \in \mathbb{P}, \; \ell \in \mathbb{N}.$$

(multiplicative fns are uniquely determined by their values on prime powers)

Then $f * g$ multiplicative and

$$(f*g)(p^\ell) = \sum_{k=0}^{\ell} f(p^k) g(p^{\ell-k}) = \sum_{k=0}^{\ell} f(p^k) f^{-1}(p^{\ell-k})$$

$$= (f * f^{-1})(p^\ell) = e(p^\ell).$$

$\Rightarrow f * g = e$, as both $f*g$ and e are determined uniquely by their values on prime powers.

By uniqueness of inverse, $g = f^{-1}$. ∎

Example: Let $\mu := \mathcal{E}^{-1}$ Möbius function.

Lemma:

$$\mu(n) = \begin{cases} 1, & n = 1 \\ (-1)^r, & n = p_1 \cdots p_r \\ 0, & \text{else.} \end{cases}$$

product of distinct primes.

Note: $|\mu| = \mu^2$ is the indicator function of square-free numbers.

Pf: $\mu(1) = 1$ ✓

$$\mu(p^l) = -\frac{1}{\varepsilon(1)} \sum_{k=1}^{l} \varepsilon(p^k)\mu(p^{l-k}) = -\sum_{k=0}^{l-1} \mu(p^k)$$

$\mu(p) = -1$ and inductively for $l \geq 2$, $\mu(p^l) = 0$.

Now $\mu = \varepsilon^{-1} \Rightarrow \mu$ is mult, so claim follows. ☒

Theorem (Möbius Inversion Formula). Let $f, g \in \mathcal{A}$.

$$g(n) = \sum_{d|n} f(d), \quad \forall n \in \mathbb{N}$$

$$\Longleftrightarrow \quad f(n) = \sum_{d|n} \mu(d)\, g\left(\frac{n}{d}\right).$$

Proof: $g = f * \varepsilon \Longleftrightarrow g * \mu = (f * \varepsilon) * \mu = f * (\varepsilon * \mu) = f.$ ☒

Exercise: $\frac{\varphi(n)}{n} = \sum_{d|n} \frac{\mu(d)}{d}$.

Summation techniques

Convolution method: A trick to compute partial sums of a Dirichlet convolution.

Main idea: (change order of summation)

Suppose $g, h \in \mathcal{A}$. Then

$$\sum_{n \le x} g * h(n) = \sum_{n \le x} \sum_{d \mid n} g(d) h\left(\frac{n}{d}\right)$$

$$= \sum_{d \le x} g(d) \sum_{\substack{n \equiv 0 (d) \\ n \le x}} h\left(\frac{n}{d}\right)$$

$d \mid n \Rightarrow d \le x$

$$= \sum_{d \le x} g(d) \sum_{m \le \frac{x}{d}} h(m).$$

Example: $\sum_{n \le x} \varphi(n) = Cx^2 + O(x \log x)$

We know $\varphi = \mu * \mathrm{id}$.

$$\Rightarrow \sum_{n \le x} \varphi(n) = \sum_{d \le x} \mu(d) \sum_{m \le \frac{x}{d}} m$$

But $\sum_{m \le \frac{x}{d}} m = \frac{1}{2}\left(\frac{x}{d}\right)^2 + O\left(\frac{x}{d}\right)$ (exercise).

$$\Rightarrow \sum_{n \le x} \varphi(n) = \sum_{d \le x} \mu(d) \cdot \left(\frac{1}{2}\left(\frac{x}{d}\right)^2 + O\left(\frac{x}{d}\right)\right).$$

$$= \frac{x^2}{2} \sum_{d \le x} \frac{\mu(d)}{d^2} + O\left(\sum_{d \le x} \mu(d) \frac{x}{d} \right). \quad (*)$$

Note that $\sum_{d=1}^{\infty} \frac{\mu(d)}{d^2}$ convergent, and

$$\sum_{d \le x} \frac{\mu(d)}{d^2} = \underbrace{\sum_{d=1}^{\infty} \frac{\mu(d)}{d^2}}_{=:C} - \sum_{d > x} \frac{\mu(d)}{d^2}$$

$$= C + O\left(\sum_{d > x} \frac{\mu(d)}{d^2} \right) = C + O\left(\sum_{d > x} \frac{1}{d^2} \right)$$

$$= C + O\left(\frac{1}{x} \right). \quad (**) \qquad \le \int_x^{\infty} \frac{1}{t^2}\,dt = \frac{1}{x}.$$

Also $\left| \sum_{d \le x} \mu(d) \frac{x}{d} \right| \le x \sum_{d \le x} \frac{1}{d} \ll x \log x$. $(***)$

From $(*)$, $(**)$, $(***)$, we get

$$\sum_{n \le x} f(n) = \frac{x^2}{2} \left(C + O\left(\frac{1}{x} \right) \right) + O(x \log x)$$

$$= x^2 \cdot \frac{C}{2} + O(x \log x).$$

Summation formulae

Proposition (Approximating sum by integral)
Let $f:[y,x]\longrightarrow \mathbb{R}$ monotonic, then

$$\sum_{y<n\le x} f(n) = \int_y^x f(t)\,dt + O(|f(x)|+|f(y)|)$$

Examples:

(i) $$\sum_{1\le n\le x} \frac{1}{n} = 1 + \sum_{1<n\le x} \frac{1}{n} =$$

$$= 1 + \int_1^x \frac{1}{t}\,dt + O\left(1+\frac{1}{x}\right)$$

$$= \log x + O(1).$$

(ii) $$\sum_{1\le n\le x} \log n = x\log x - x + O(\log x).$$

(iii) $$\sum_{n\le x} n = \frac{x^2}{2} + O(x).$$

Proof of proposition:

Case 1: $y, x \in \mathbb{Z}$, f increasing.

We use that $$\int_{n-1}^{n} f(t)\,dt \le f(n) \le \int_{n}^{n+1} f(t)\,dt$$

Hence $\displaystyle\int_y^x f(t)\,dt \le \sum_{y<n\le x} f(n) \le f(x) + \int_{y+1}^x f(t)\,dt.$

$$\Rightarrow 0 \le \sum_{y<n\le x} f(n) - \int_y^x f(t)\,dt$$

$$= -\int_y^{y+1} f(t)\,dt + \left(\sum_{y<n\le x} f(n) - \int_{y+1}^x f(t)\,dt\right)$$

$$\le \left|\int_y^{y+1} f(t)\,dt\right| + f(x)$$

$$\le 2(|f(x)| + |f(y)|).$$

(since $|f(t)| \le \max\{|f(x)|, |f(y)|\}$ for $t \in [y,x]$) ✓

Case 2: y, x general, f increasing.

From previous case,

$$\sum_{y<n\le x} f(n) = \int_{\lfloor y\rfloor+1}^{\lfloor x\rfloor} f(t)\,dt + O(|f(\lfloor y\rfloor+1)| + |f(\lfloor x\rfloor)|)$$

$$= \int_y^x f(t)\,dt + O\left(\left|\int_y^{\lfloor y\rfloor+1} f(t)\,dt\right| + \left|\int_{\lfloor x\rfloor}^x f(t)\,dt\right|\right)$$

$$+ O(|f(y)| + |f(x)|)$$

$$= \int_y^x f(t)\,dt + O(|f(y)| + |f(x)|) \checkmark$$

Case 3: y, x general, f decreasing

Let $F = -f$, so F increasing, so

$$\sum_{y<n\leq x} f(n) = -\sum_{y<n\leq x} F(n) = -\int_y^x F(t)\,dt + O(|F(x)| + |F(y)|)$$

$$= \int_y^x f(t)\,dt + O(|f(x)| + |f(y)|).$$

☒

Sometimes we have better approximations for general class of funtions:

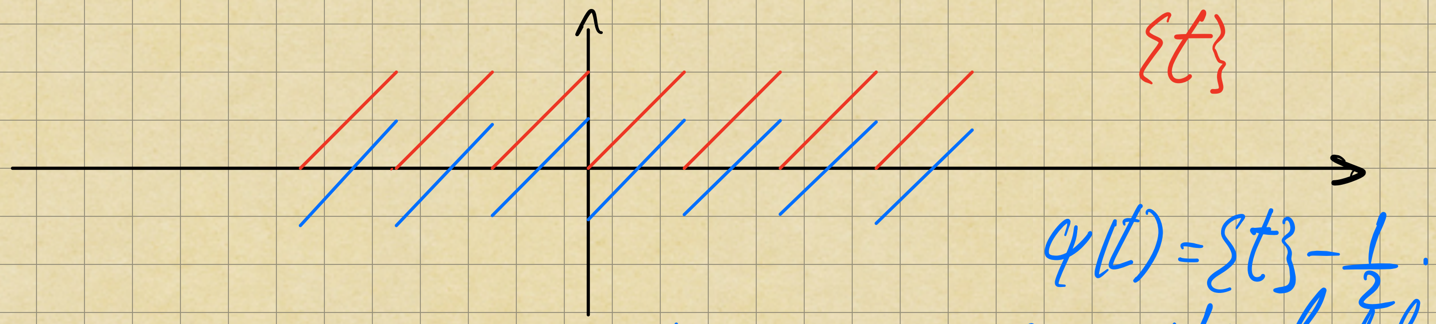
Theorem (Euler-Maclaurin formula).
Let $f : [y, x] \to \mathbb{C}$ a C^1-function. Then

$$\sum_{y<n\leq x} f(n) = \int_y^x f(t)\,dt + \int_y^x f'(t)\psi(t)\,dt + f(y)\psi(y) - f(x)\psi(x),$$

where $\psi(t) = \{t\} - \frac{1}{2}$.

↗ fractional part

Exact formula!



has mean 0 over every interval of length 1.

Proof:

Claim: Let $n \in \mathbb{Z}$. Then

$$\frac{f(n)+f(n+1)}{2} = \int_n^{n+1} f(t)\,dt + \int_n^{n+1} \psi(t) f'(t)\,dt.$$

Proof of claim:

$$\int_n^{n+1} \psi(t) f'(t)\,dt = \int_n^{n+1} \left(t-n-\frac{1}{2}\right) f'(t)\,dt$$

$$= \int_n^{n+1} t f'(t)\,dt - \left(n+\frac{1}{2}\right) \int_n^{n+1} f'(t)\,dt$$

$$= \Big[t f(t)\Big]_n^{n+1} - \int_n^{n+1} f(t)\,dt - \left(n+\frac{1}{2}\right) \Big[f(t)\Big]_n^{n+1}$$

$$= (n+1) f(n+1) - n f(n) - \int_n^{n+1} f(t)\,dt - \left(n+\frac{1}{2}\right)\left(f(n+1) - f(n)\right)$$

$$= \frac{f(n+1)}{2} + \frac{f(n)}{2} - \int_n^{n+1} f(t)\,dt. \quad \checkmark$$

Case 1: y, x integers. We sum the quantities in the claim for all integers $y \le n < x$.

We get $\frac{f(y)}{2} + \sum_{y<n<x} f(n) + \frac{f(x)}{2} = \int_y^x f(t)dt + \int_y^x \psi(t) f'(t)$

$$\Rightarrow \sum_{y<n\le x} f(n) = \int_y^x f(t)\,dt + \int_y^x \psi(t) f'(t) + \frac{f(x)}{2} - \frac{f(y)}{2}$$

Claim follows from the fact that $\psi(\kappa) = -\frac{1}{2}$, for $\kappa \in \mathbb{Z}$.

Case 2: General y, x.

Exercise: $\int_y^{\lfloor y\rfloor+1} \psi(t) f'(t)dt = -\psi(y) f(y) + \frac{f(\lfloor y\rfloor+1)}{2} - \int_y^{\lfloor y\rfloor+1} f(t)dt$

$$\int_{\lfloor x\rfloor}^x \psi(t) f'(t) = \psi(x) f(x) + \frac{f(\lfloor x\rfloor)}{2} - \int_{\lfloor x\rfloor}^x f(t)dt.$$

☒

Application: $\sum_{1\le n\le x} \frac{1}{n} = \log x + \gamma + O\left(\frac{1}{x}\right)$.
γ ↳ Euler constant

Recall: Monotonicity only gave us $\log x + O(1)$.

By Euler-Maclaurin: $\sum_{1\le n\le x} \frac{1}{n} = 1 + \sum_{1<n\le x} \frac{1}{n}$

$$= 1 + \int_1^x \left(\frac{1}{t} - \frac{\psi(t)}{t^2}\right)dt - \frac{1}{2} - \frac{\psi(x)}{x}$$

$$= \log x + \underbrace{\left(\frac{1}{2} - \int_1^\infty \frac{\psi(t)}{t^2}\,dt\right)}_{\text{This is the constant } \gamma.} + \int_x^\infty \frac{\psi(t)}{t^2}\,dt - \underbrace{\frac{\psi(x)}{x}}_{O\left(\frac{1}{x}\right)}$$

Now $\left| \int_x^\infty \frac{\psi(t)}{t^2}\,dt \right| \ll \int_x^\infty \frac{1}{t^2}\,dt = \frac{1}{x} = O\left(\frac{1}{x}\right).$

Theorem (Abel summation or Partial summation)

Let $f \in \mathcal{R}$, $0 < y < x$, $g \in C^1([y,x])$.

Define $F(T) := \sum_{n \le T} f(n)$.

Then $$\sum_{y < n \le x} f(n) g(n) = F(x)g(x) - F(y)g(y) - \int_y^x F(t) g'(t)\,dt$$

Proof: WLOG $\lfloor y \rfloor + 1 \le \lfloor x \rfloor$

(otherwise no integer between y and x, LHS $= 0$
and $F(x) = F(y) = F(t)$, $\forall t \in [y,x]$,
so RHS $= 0$).

Then $$\sum_{y < n \le x} f(n) g(n) = \sum_{\lfloor y \rfloor + 1 \le n \le \lfloor x \rfloor} (F(n) - F(n-1))\, g(n)$$

$$= \sum_{\lfloor y \rfloor + 1 \le n \le \lfloor x \rfloor} F(n)\, g(n) - \sum_{\lfloor y \rfloor \le n \le \lfloor x \rfloor - 1} F(n)\, g(n+1)$$

$$= F(x)\,g(\lfloor x\rfloor) - F(y)\,g(\lfloor y\rfloor + 1) - \sum_{\lfloor y\rfloor+1 \le n \le \lfloor x\rfloor - 1} \underbrace{F(n)\big(g(n+1) - g(n)\big)}_{\int_n^{n+1} F(\xi)\,g'(\xi)\,d\xi}.$$

$$= F(x)\,g(\lfloor x\rfloor) - F(y)\,g(\lfloor y\rfloor+1) - \int_{\lfloor y\rfloor+1}^{\lfloor x\rfloor} F(t)\,g'(t)\,dt$$

Now we see $\int_{\lfloor x\rfloor}^{x} F(t)\,g'(t)\,dt = F(x)\big(g(x) - g(\lfloor x\rfloor)\big)$.

$$\int_{y}^{\lfloor y\rfloor+1} F(t)\,g'(t)\,dt = F(y)\big(g(\lfloor y\rfloor+1) - g(y)\big).$$

$\blacksquare$

<u>Corollary</u>: Let $g : [y, \infty) \to \mathbb{C}$, $f \in \mathcal{A}$ and suppose $F(x)\,g(x) \to 0$ as $x \to \infty$.

Then $$\sum_{n \ge y} f(n)\,g(n) = -\int_{y}^{\infty} F(t)\,g'(t)\,dt - F(y)\,g(y)$$

whenever both sides converge.

Example: $\exists\, C' > 0$ s.t. $\displaystyle\sum_{n\le x} \frac{\varphi(n)}{n} = C'x + O((\log x)^2)$.

Pf: Denote $F(y) = \displaystyle\sum_{n \le y} \varphi(n)$. By earlier today,

$$F(y) = C\cdot y^2 + O(y \log y).$$

Hence by partial summation,

$$\sum_{n\le x} \frac{\varphi(n)}{n} = 1 + \frac{F(x)}{x} - 1 + \int_1^x F(t)\cdot\frac{1}{t^2}\,dt$$

$$= \frac{Cx^2 + O(x\log x)}{x} + \int_1^x \frac{C\cdot t^2 + O(t\log t)}{t^2}\,dt$$

$$= C\cdot x + O(\log x) + C(x-1) + O\left(\int_1^x \frac{\log t}{t}\,dt\right)$$

$$= 2\cdot C\cdot x + O(\log x) + O\left([(\log t)^2]_1^x\right)$$

$$= 2\cdot C\cdot x + O((\log x)^2). \quad \blacksquare$$